University of KwaZulu-Natal

Discipline of Electrical, Electronic and Computer Engineering

ENEL 4CS: Control Systems II **Main Examination Paper**

Time Allowed: 2hrs **Total Marks: 100** **Date: 25th May, 2026**

Examiner: Prof. J.T. Agee **Moderator: Mr. B. Naido** **External Examiner: Dr. Richard Chizonga**

Instructions: *This paper has four questions spread over three pages. Answer all four questions. A two-page formula sheet is attached at the end of the paper. You may write your answers in blue or black ink. Do not write in pencil or any colour of ink other than the two advised, as such work will not be graded. Plots not made using the appropriate graph sheets will not be marked.*

No programmable calculators allowed during this examination.

Provided: One Bode Charts and one graph paper**.**

QUESTION ONE [25 Mark]

You are required to use a mathematically based-frequency method to design a PI controller $G_{PI}(s) = K_P + \frac{K_i}{s}$ for the control system, whose transfer function is given as $G_1(s) = \frac{50}{(s+5)(s^2+5s+20)}$. You are not required to make any frequency plots in this design problem.

1.1 Design the PI controller such that the controlled system has a damping factor of $\xi = 0.5$ at the frequency of $\omega = 5.03\ rads^{-1}$. [17]

1.2 Using the Tustin's approximation, convert the controller transfer function $G_{PI}(s)$ into a discrete-time transfer function where the system is sampled at a rate of T=0.1 sec. [3].

1.3 Show how the Discrete-time version of $G_{PI}(s)$ could be implemented using a micro-controller. [5]

QUESTION TWO [25 Marks]

Using a Bode plot-based approach, design a lead controller for the third-order control system whose transfer function is given by $G_2(s) = \frac{K}{(s+1)(s+8)(s+2)}$: such that the performance of the controlled system meets the following two requirements:

(i) A steady-state error of 0.01.
(ii) A phase margin of 14^0.

Adjust your final phase margin calculation by 5^0.

QUESTION THREE [19 Marks]

In Figure 1, the transfer function $G_{PF}(s)$ is the pre-filter, and $G_3(s) = \frac{320}{(s+b)^2}$ is the system to be controlled. The b-parameter in $G_3(s)$ varies in the range $-1.28 < b \le 7$. Variations in the value of the parameter ***b*** leads to changes in the poles of $G_3(s)$, with associated variations in the step response of $G_3(s)$. Furthermore, $G_{L(s)} = 2.964\frac{(s+7.783)}{(s+23.07)}$ is a lead controller designed to improve the phase margin of the system, when ***b***=3. The combination of the Robust controller $G_R(s)$ and the pre-filter $G_{PF}(s)$ is required to minimise deviation in the response of the overall system, when **b** deviates from the nominal design value.

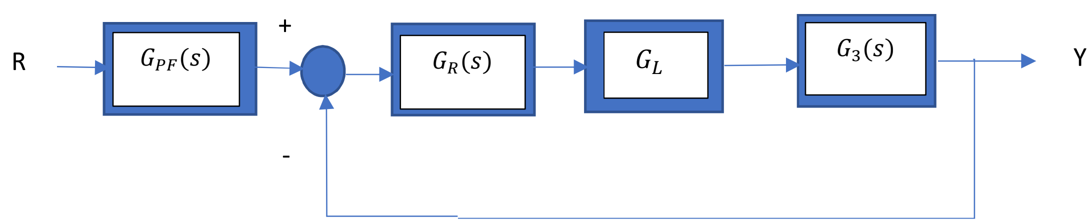

Figure 1: Robust controller design

You are required to design the robust controller and the pre-filter using a root-locus based approach:

3.1 Evaluate the product transfer function $G_4(s) = G_L(s)G_3(s)$. Hence obtain the closed-loop transfer function T(s), of the system in Figure 1, when $G_{PF}(s) = G_R(s) = 1$. [4]

3.2 Evaluate the poles of T(s). [2]

3.3 Draw a detailed root-locus plot of $G_4(s)$, showing all poles and zeros of $G_4(s)$, the asymptotes, asymptote angles, break-away point(s). [9]

3.4 Using the root-locus plot of $G_4(s)$, derive the transfer function of the robust controller $G_R(s)$, and that of the pre-filter. [4]

QUESTION FOUR [31 Marks]

Figure 2 shows a third-order DC motor system for the control of an angular position $\theta(t)$. The direction of rotation of the motor is determined by the switched position of the ideal-relay nonlinearity. The relay switches between the voltage values of $\pm U$. The input applied to the relay is x(t), whilst the output of the relay is $y(t)$. The command input into the overall DC motor control system is set at $\theta_d = 0$.

4.1 If the input into the nonlinearity is given by $x(t) = X \sin \omega t\,; X_0 > 0$, sketch the output y(t) of the relay nonlinearity for one period of the input *x(t)*. Hence, write the expression for the output $y(t)$ of the nonlinearity for one period of the input waveform. [8]

4.2 Derive the describing function $N(X, \omega)$ of the relay nonlinearity. [10]

4.3 Using the Nyquist stability criterion, determine the amplitude X of any limit circle(s) that may exist within the DC motor control system. [13]

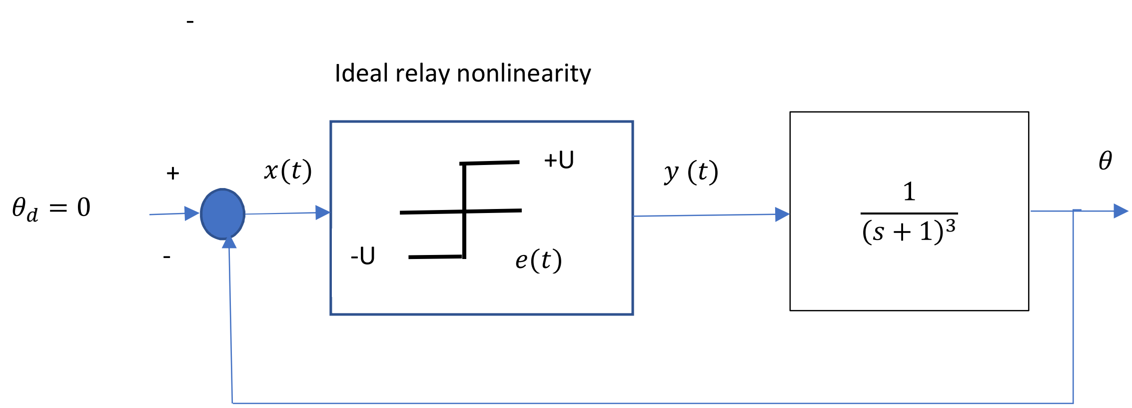

Figure 2: A DC motor control system with dead-zone nonlinearity.

ENEL 4CS Formula Sheet

Modeling

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \quad (2.97); \qquad \frac{V_o(s)}{V_i(s)} = \frac{Z_1(s)+Z_2(s)}{Z_1(s)} \quad (2.104)$$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2} \quad (2.133); \qquad \frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1} \quad (2.135)$$

$$\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2 \quad \text{(see after 2.138)}$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_aJ_m)}{s\left[s+\frac{1}{J_m}\left(D_m+\frac{K_tK_b}{R_a}\right)\right]} \quad (2.153)$$

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} \quad (2.162); \qquad K_b = \frac{e_a}{\omega_{\text{no-load}}} \quad (2.163)$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \quad (3.73)$$

Time Response

$$T_r = \frac{2.2}{a} \quad (4.9); \qquad T_s = \frac{4}{a} \quad (4.10)$$

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad (4.22)$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 \quad (4.38)$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}} \quad (4.39)$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad (4.34); \qquad T_s = \frac{4}{\zeta\omega_n} \quad (4.42)$$

Steady-State Error

$$e(\infty) = e_{\text{step}}(\infty) = \frac{1}{1+\lim_{s\to 0} G(s)} \quad (7.30); \qquad K_p = \lim_{s\to 0} G(s) \quad (7.33)$$

$$e(\infty) = e_{\text{ramp}}(\infty) = \frac{1}{\lim_{s\to 0} sG(s)} \quad (7.31); \qquad K_v = \lim_{s\to 0} sG(s) \quad (7.34)$$

$$e(\infty) = e_{\text{parabola}}(\infty) = \frac{1}{\lim_{s\to 0} s^2G(s)} \quad (7.32); \qquad K_a = \lim_{s\to 0} s^2G(s) \quad (7.35)$$

Root Locus

$$\angle KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ \quad (8.13)$$

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{ finite poles} - \#\text{ finite zeros}} \quad (8.27)$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{ finite poles} - \#\text{ finite zeros}} \quad (8.28)$$

$$\theta = \sum \text{finite zero angles} - \sum \text{finite pole angles}$$

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{finite pole lengths}}{\prod \text{finite zero lengths}} \quad (8.51)$$

Frequency Response

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad (10.52); \qquad \omega_p = \omega_n\sqrt{1-2\zeta^2} \quad (10.53)$$

$$\omega_{\text{BW}} = \omega_n\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \quad (10.54)$$

$$\Phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}} \quad (10.73)$$

$$\phi_{\max} = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} = \sin^{-1}\frac{1-\beta}{1+\beta} \quad (11.11)$$

$$\omega_{\max} = \frac{1}{T\sqrt{\beta}} \quad (11.9); \qquad |G_c(j\omega_{\max})| = \frac{1}{\sqrt{\beta}} \quad (11.12)$$

State Space

$$\mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}] \quad (12.26)$$

$$\dot{\mathbf{x}} = (\mathbf{A}-\mathbf{BK})\mathbf{x}+\mathbf{B}r; \quad y = \mathbf{Cx} \quad (12.3); \qquad \mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad (12.79)$$

$$\dot{\mathbf{e}}_\mathbf{x} = (\mathbf{A}-\mathbf{LC})\mathbf{e}_\mathbf{x}; \quad y-\hat{y} = \mathbf{Ce}_\mathbf{x} \quad (12.64)$$

Digital Control

$$e^*(\infty) = \lim_{z\to 1}(1-z^{-1})E(z) \quad (13.66)$$

$$K_p = \lim_{z\to 1} G(z) \quad (13.70); \qquad K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) \quad (13.73)$$

$$K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2G(z) \quad (13.75)$$

Also: $\theta_P = sin^{-1}(\frac{1}{M_P})$; Radius of M-circle $=\left|\frac{M_P}{M_P^2-1}\right|$

Centre of M-circle$=(\frac{-M^2}{M^2-1}, 0)$

$$T_r = \frac{1}{\omega_n}\left\{1.76\xi^3 - 0.417\xi^2 + 1.039\varsigma + 1\right\}$$

$$s_1, s_2 = -\xi\omega_n + \omega_n\sqrt{1-\xi^2}$$

Lag control: $G_C(s) = K_C\left(\frac{s+1/T}{s+1/\alpha T}\right); \alpha > 1, K_C = \frac{1/\alpha T}{1/T} = \frac{1}{\alpha}$

Lead Control:

$$G_C(s) = \frac{1}{\beta}\left(\frac{s+1/T}{s+1/\beta T}\right); \beta < 1; \qquad G_C(j\omega_{\max}) = \frac{1}{\sqrt{\beta}}$$

$$\omega_{\max} = \frac{1}{T\sqrt{\beta}}; \phi_{\max} = \sin^{-1}\left\{\frac{1-\beta}{1+\beta}\right\}, \beta = \frac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}}$$

Table 1: Formulae for the Nichols-Ziegler's First Method of Controller Design

Controller	**Controller Gains**		
	K_C	T_i	T_d
P	K_0		
PI	$0.9K_0$	$3.3\tau_{dead}$	
PID	$1.2K_0$	$2\tau_{dead}$	$0.5\tau_{dead}$

Table 2: Nichols-Zeigler's Second Method of Controller Design Based on the Critical Gain Parameter K_{Cr}

Type of Controller	K_P	T_i	T_d
P	$0.5K_{cr}$	∞	**0**
PI	$0.45K_{cr}$	$P_{Cr}/1.2$	**0**
PID	$0.6K_{cr}$	$0.5P_{Cr}$	$0.125P_{Cr}$

Table 3: A Frequency-Based Computational Approach to PI, PD and PID Formulae

Type of Controller	K_P	K_i	K_d
PD	$cos\theta/A_1$		$sin\theta/\omega_1 A_1$
PI	$cos\theta/A_1$	$-\omega_1 sin\theta/A_1$	
PID	$cos\theta/A_1$	**K_i is evaluated to meet the steady-state error requirement**	$\frac{K_i}{\omega_1^2} + \frac{sin\theta}{\omega_1 A_1}$

Table 4: Tustin Transformations (Approximations)

$s = \frac{2}{T}\frac{(z-1)}{(z+1)}$	$z = \frac{-(s+\frac{2}{T}}{(s-\frac{2}{T})} = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$