UNIVERSITY OF
KWAZULU-NATAL

University of KwaZulu-Natal

School of Engineering

Electrical, Electronic & Computer Engineering

MAIN EXAMINATION–2025

ENEL2FT: FIELD THEORY

Time allowed: 2 hours **Max. Marks: 70**

Instructions to Candidates:

1. This paper contains 7 questions.
2. Answer all **SEVEN** questions.
3. The marks for each question are indicated.
4. Answers should show sufficient working steps to indicate the solution method used.
5. The notations and symbols have their usual meaning.

The following materials are provided:

Vector Formula Sheet and Mathematical Tables

Examiner
Prof P Kumar
Moderator
Dr S Rezenom

Q1. (a) Consider the point charges of $Q_1 = 10\ mC$ and $Q_2 = 20\ mC$ located at $A(1,2,3)$ and $B(3,2,1)$, respectively. Determine the electric field intensity $\vec{E}$ at $C(-3,-1,2)$. [6]

(b) Consider an infinite line charge of $\rho_l = 8\ mC/m$ on the x-axis. Determine the electric field intensity $\vec{E}$ at $P(4,5,7)$. [4]

Q2. A charge distribution in free space has $\rho_v = 2r\ {}^{nC}/_{m^3}$ for $0 \leq r \leq 10\ m$ and zero otherwise. Determine:

(a) the electric field intensity $\vec{E}$ at P(2,90°,60°), [4]

(b) the electric field intensity $\vec{E}$ at Q(7,8,$\sqrt{31}$), [5]

(c) the electric potential V at P(2,90°,60°). [1]

Q3. The surface $x = 0$ separates two perfect dielectrics. For $x > 0$ let $\varepsilon_r = \varepsilon_{r1} = 3$, while $\varepsilon_{r2} = 5$ for $x < 0$. If $\overrightarrow{E_1} = 80\hat{x} - 60\hat{y} - 30\hat{z}$ V/m. Determine:

(a) the electric flux density $\overrightarrow{D_2}$ for $x < 0$, [5]

(b) the polarization vectors $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$, [3]

(c) the angle θ_1 between the electric field intensity $\overrightarrow{E_1}$ and the interface. [2]

Q4. Concentric conducting spheres are located at $r = 5\ mm$ and $r = 20\ mm$. The region between the spheres is filled with a perfect dielectric with the dielectric constant of $\varepsilon_r = 4$. If the inner sphere is at $100\ V$ and the outer sphere is at $0\ V$, determine:

(a) the expression for the voltage V between the spheres, [6]

(b) the maximum value of the electric field intensity $\vec{E}$, [2]

(c) the location of the $20\ V$ and $40\ V$ equipotential surfaces. [2]

Q5. (a) An infinite long current carrying conductor lies on the y-axis and carries a current of 2 A in the -$\hat{y}$ direction. Determine the magnetic field intensity $\vec{H}$ at:

(i) A(2,3,0), [3]

(ii) B(0,12,-4). [3]

(b) Calculate the current density $\vec{J}$ at A(2,3,4) if $\vec{H} = x^2z\hat{y} - y^2x\hat{z}$ A/m. [4]

Q6. A unit normal vector from region-2 ($\mu = 2\mu_0$) to region-1 ($\mu = \mu_0$) is $\widehat{n_{21}} = \frac{6\hat{x}+2\hat{y}-3\hat{z}}{7}$. If $\overrightarrow{H_1} = 10\hat{x} + \hat{y} + 12\hat{z}$ A/m, determine:

(a) the tangential and normal components of $\overrightarrow{H_1}$, [4]

(b) the magnetic flux density in region-2 ($\overrightarrow{B_2}$). [6]

Q7. The iron core shown in Fig. Q7 has a cross section of 2 $cm \times 2\ cm$ and a relative permeability of 1000. The coil has 500 turns and carries a current of i = 2A. Find:

(a) the reluctance of the centre limb, [2]

(b) the reluctance of the air gap 'Gap a', [2]

(c) the flux in the centre limb. [6]

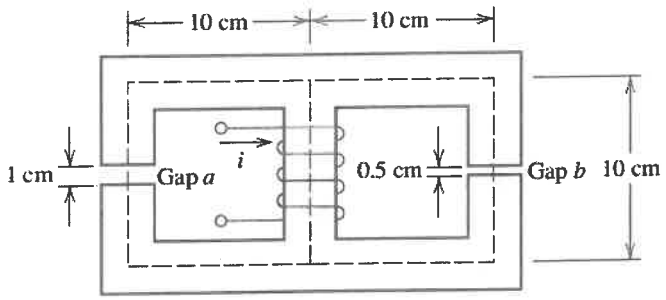


Fig. Q7

VECTOR OPERATORS AND SOME PHYSICAL CONSTANTS

Coordinate Transformations

Rectangular to Cylindrical:

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} cos\emptyset & sin\emptyset & 0 \\ -sin\emptyset & cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Cylindrical to Rectangular:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} cos\emptyset & -sin\emptyset & 0 \\ sin\emptyset & cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix}$$

Rectangular to Spherical:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix} = \begin{bmatrix} sin\theta cos\emptyset & sin\theta sin\emptyset & cos\theta \\ cos\theta cos\emptyset & cos\theta sin\emptyset & -sin\theta \\ -sin\emptyset & cos\emptyset & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Spherical to Rectangular:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} sin\theta cos\emptyset & cos\theta cos\emptyset & -sin\emptyset \\ sin\theta sin\emptyset & cos\theta sin\emptyset & cos\emptyset \\ cos\theta & -sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix}$$

Cylindrical to Spherical:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix} = \begin{bmatrix} sin\theta & 0 & cos\theta \\ cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix}$$

Differential Length, Area and Volume

Rectangular Coordinates:

$$\vec{dl} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$\vec{dS} = dydz\hat{x} + dxdz\hat{y} + dxdy\hat{z}$$

$$dv = dxdydz$$

Cylindrical Coordinates:

$$\vec{dl} = d\rho\hat{\rho} + \rho d\emptyset\hat{\emptyset} + dz\hat{z}$$

$$\vec{dS} = \rho d\emptyset dz\hat{\rho};\ \vec{dS} = d\rho dz\hat{\emptyset};\ \vec{dS} = \rho d\emptyset d\rho\hat{z}$$

$$dv = \rho d\rho d\emptyset dz$$

Spherical Coordinates:

$$\vec{dl} = dr\hat{r} + rd\theta\hat{\theta} + rsin\theta d\emptyset\hat{\emptyset}$$

$$\vec{dS} = r^2 sin\theta d\theta d\emptyset\hat{r};$$

$$\vec{dS} = rsin\theta dr d\emptyset\hat{\theta};$$

$$\vec{dS} = rdrd\theta\hat{\emptyset};\ dv = r^2 sin\theta dr d\theta d\emptyset$$

Some Physical Constants

Permittivity of free space:

$$\varepsilon_0 = 8.854 \times 10^{-12}\ F/m$$

Permeability of free space:

$$\mu_0 = 4\pi \times 10^{-7}\ H/m$$

Velocity of light in free space:

$$c = 2.988 \times 10^{8}\ m/s$$

Electron charge:

$$e = 1.6 \times 10^{-19}\ C$$

Vector Differential Operators

Rectangular Coordinates:

$$\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$

$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical Coordinates:

$$\vec{\nabla}V = \frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \emptyset}\hat{\emptyset} + \frac{\partial V}{\partial z}\hat{z}$$

$$\vec{\nabla}.\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\emptyset}{\partial \emptyset} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\emptyset} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\emptyset & A_z \end{vmatrix}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \emptyset^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical Coordinates:

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{rsin\theta}\frac{\partial V}{\partial \emptyset}\hat{\emptyset}$$

$$\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{rsin\theta}\frac{\partial}{\partial \theta}(A_\theta sin\theta) + \frac{1}{rsin\theta}\frac{\partial A_\emptyset}{\partial \emptyset}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & rsin\theta\hat{\emptyset} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ A_r & rA_\theta & rsin\theta A_\emptyset \end{vmatrix}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \emptyset^2}$$

VECTOR FORMULAS

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \tag{1-29}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{1-30}$$

$$\nabla\times\nabla u = 0 \tag{1-48}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \tag{1-49}$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \tag{1-106}$$

$$\frac{d}{d\sigma}(u\mathbf{A}) = \frac{du}{d\sigma}\mathbf{A} + u\frac{d\mathbf{A}}{d\sigma} \tag{1-107}$$

$$\frac{d}{d\sigma}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{d\sigma}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{d\sigma} \tag{1-108}$$

$$\frac{d}{d\sigma}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{d\sigma}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{d\sigma} \tag{1-109}$$

$$\nabla(u+v) = \nabla u + \nabla v \tag{1-110}$$

$$\nabla(uv) = u\nabla v + v\nabla u \tag{1-111}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{1-112}$$

$$\nabla(\mathbf{C}\cdot\mathbf{r}) = \mathbf{C} \qquad \text{where } \mathbf{C} = \text{const.} \tag{1-113}$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B} \tag{1-114}$$

$$\nabla\cdot(u\mathbf{A}) = \mathbf{A}\cdot(\nabla u) + u(\nabla\cdot\mathbf{A}) \tag{1-115}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \tag{1-116}$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B} \tag{1-117}$$

$$\nabla\times(u\mathbf{A}) = (\nabla u)\times\mathbf{A} + u(\nabla\times\mathbf{A}) \tag{1-118}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{1-119}$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{1-120}$$

where

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right) + \hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right) \tag{1-121}$$

VECTOR OPERATIONS

RECTANGULAR COORDINATES

$$\nabla u = \hat{\mathbf{x}}\frac{\partial u}{\partial x} + \hat{\mathbf{y}}\frac{\partial u}{\partial y} + \hat{\mathbf{z}}\frac{\partial u}{\partial z} \tag{1-37}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{1-42}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{1-43}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{1-46}$$

CYLINDRICAL COORDINATES

$$\nabla u = \hat{\boldsymbol{\rho}}\frac{\partial u}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{\mathbf{z}}\frac{\partial u}{\partial z} \tag{1-85}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \tag{1-87}$$

$$\nabla\times\mathbf{A} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{\mathbf{z}}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \varphi}\right] \tag{1-88}$$

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} \tag{1-89}$$

SPHERICAL COORDINATES

$$\nabla u = \hat{\mathbf{r}}\frac{\partial u}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial u}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi} \tag{1-101}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi} \tag{1-103}$$

$$\nabla\times\mathbf{A} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial \varphi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right] + \frac{\hat{\boldsymbol{\varphi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right] \tag{1-104}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} \tag{1-105}$$

Useful mathematical tables

C.1 A brief list of series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots |x| < 1$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots |x| < 1$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots |x| < 1$$

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots = \infty$$

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \ln(2)$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad \text{for all } x$$

C.2 A list of trigonometric identities

$$e^{\theta} = \cosh(\theta) + \sinh(\theta) = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad \text{where } j = \sqrt{-1}$$

$$\cosh(\theta) = \tfrac{1}{2}[e^{\theta} + e^{-\theta}]$$

$$\sinh(\theta) = \tfrac{1}{2}[e^{\theta} - e^{-\theta}]$$
$$\cos(\theta) = \tfrac{1}{2}[e^{j\theta} + e^{-j\theta}]$$
$$\sin(\theta) = \tfrac{1}{2j}[e^{j\theta} - e^{-j\theta}]$$
$$\sin(-\alpha) = -\sin(\alpha) \qquad \sin(\alpha) = \cos(\alpha - \pi/2)$$
$$\cos(-\alpha) = \cos(\alpha) \qquad \cos(\alpha) = -\sin(\alpha - \pi/2)$$
$$\cosh(j\alpha) = \cos(\alpha)$$
$$\sinh(j\alpha) = j\sin(\alpha)$$
$$\cos(j\beta) = \cosh(\beta)$$
$$\sin(j\beta) = j\sinh(\beta)$$
$$\sinh(\alpha + \beta) = \sinh(\alpha)\cosh(\beta) + \cosh(\alpha)\sinh(\beta)$$
$$\cosh(\alpha + \beta) = \cosh(\alpha)\cosh(\beta) + \sinh(\alpha)\sinh(\beta)$$
$$\sinh(\alpha + j\beta) = \sinh(\alpha)\cos(\beta) + j\cosh(\alpha)\sin(\beta)$$
$$\cosh(\alpha + j\beta) = \cosh(\alpha)\cos(\beta) + j\sinh(\alpha)\sin(\beta)$$
$$\sin(\alpha + j\beta) = \sin(\alpha)\cosh(\beta) + j\cos(\alpha)\sinh(\beta)$$
$$\sin(\alpha - j\beta) = \sin(\alpha)\cosh(\beta) - j\cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + j\beta) = \cos(\alpha)\cosh(\beta) - j\sin(\alpha)\sinh(\beta)$$
$$\cos(\alpha - j\beta) = \cos(\alpha)\cosh(\beta) + j\sin(\alpha)\sinh(\beta)$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$
$$\sin(3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha)$$
$$\begin{aligned}\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha)\\ &= 2\cos^2(\alpha) - 1\\ &= 1 - 2\sin^2(\alpha)\end{aligned}$$
$$\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$$
$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$
$$1 + \tan^2(\alpha) = \sec^2(\alpha) \qquad 1 + \cot^2(\alpha) = \csc^2(\alpha)$$
$$\sin^2(\alpha) = \tfrac{1}{2}(1 - \cos(2\alpha))$$
$$\cos^2(\alpha) = \tfrac{1}{2}(1 + \cos(2\alpha))$$
$$\sin^3(\alpha) = \tfrac{1}{4}(3\sin(\alpha) - \sin(3\alpha))$$
$$\cos^3(\alpha) = \tfrac{1}{4}(3\cos(\alpha) + \cos(3\alpha))$$
$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta)$$
$$2\cos(\alpha)\cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$
$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$
$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

C.3 A list of indefinite integrals

In the list of integrals that follows, C is simply a constant of integration.

Let $X = \sqrt{a^2 + x^2}$

$$\int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$$

$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$$

$$\int X\,dx = \frac{1}{2}x\,X + \frac{a^2}{2}\ln|x + X| + C$$

$$\int xX\,dx = \frac{1}{3}X^3 + C$$

$$\int \frac{dx}{X} = \ln[x + X] + C$$

$$\int \frac{dx}{X^3} = \frac{1}{a^2}\frac{x}{X} + C$$

$$\int \frac{dx}{X^5} = \frac{1}{a^4}\left[\frac{x}{X} - \frac{1}{3}\frac{x^3}{X^3}\right] + C$$

$$\int \frac{x\,dx}{X} = X + C$$

$$\int \frac{x\,dx}{X^3} = -\frac{1}{X} + C$$

$$\int \frac{x\,dx}{X^5} = -\frac{1}{3X^3} + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(x/a) + C$$

$$\int \frac{dx}{(a^2 + x^2)^2} = \frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3}\tan^{-1}(x/a) + C$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln|a^2 + x^2| + C$$

$$\int \frac{x\,dx}{(a^2 + x^2)^2} = -\frac{1}{2(a^2 + x^2)} + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln|(a + x)/(a - x)| + C = \frac{1}{a}\tanh^{-1}(x/a) + C$$

$$\int \frac{x\,dx}{(a^2 - x^2)} = -\frac{1}{2}\ln|a^2 - x^2| + C$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C$$

$$\int \sin(ax)\cos(bx)\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)}, \quad a \neq \pm b$$

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a \neq \pm b$$

$$\int \cos(ax)\cos(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \quad a \neq \pm b$$

$$\int \sin(ax)\cos(ax)\,dx = -\frac{\cos(2ax)}{4a} + C$$

$$\int \sin^n(ax)\cos(ax)\,dx = \frac{\sin^{n+1}(ax)}{(n+1)a} + C, \quad n \neq -1$$

$$\int \tan(ax)\,dx = -\frac{1}{a}\ln|\cos(ax)| + C$$

$$\int \cot(ax)\,dx = \frac{1}{a}\ln|\sin(ax)| + C$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) + C$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int \tan^2(ax)\,dx = \frac{1}{a}\tan(ax) - x + C$$

$$\int \cot^2(ax)\,dx = -\frac{1}{a}\cot(ax) - x + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int b^{ax}\,dx = \frac{1}{a\ln(b)}b^{ax} + C$$

$$\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)] + C$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)] + C$$

$$\int \ln(ax)\,dx = x\ln(ax) - x + C$$

$$\int x^n\ln(ax)\,dx = \frac{x^{n+1}}{n+1}\ln(ax) - \frac{x^{n+1}}{(n+1)^2} + C \qquad n \neq -1$$

$$\int \frac{1}{x}\ln(ax)\,dx = \frac{1}{2}[\ln(ax)]^2 + C$$

$$\int \sinh(ax)\,dx = \frac{1}{a}\cosh(ax) + C$$

$$\int \cosh(ax)\,dx = \frac{1}{a}\sinh(ax) + C$$
$$\int \tanh(ax)\,dx = \frac{1}{a}\ln[\cosh(ax)] + C$$
$$\int \coth(ax)\,dx = \frac{1}{a}\ln|\sinh(ax)| + C$$
$$\int \operatorname{sech}(ax)\,dx = \frac{1}{a}\sin^{-1}[\tanh(ax)] + C$$
$$\int \operatorname{csch}(ax)\,dx = \frac{1}{a}\ln|\tanh(ax/2)| + C$$
$$\int \sinh^2(ax)\,dx = \frac{\sinh(2ax)}{4a} - \frac{x}{2} + C$$
$$\int \cosh^2(ax)\,dx = \frac{\sinh(2ax)}{4a} + \frac{x}{2} + C$$
$$\int \tanh^2(ax) = x - \frac{1}{a}\tanh(ax) + C$$
$$\int \coth^2(ax)\,dx = x - \frac{1}{a}\coth(ax) + C$$
$$\int \operatorname{sech}^2(ax)\,dx = \frac{1}{a}\tanh(ax) + C$$
$$\int \operatorname{csch}^2(ax)\,dx = -\frac{1}{a}\coth(ax) + C$$

C.4 A partial list of definite integrals

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} \qquad (a > 0)$$
$$\int_0^\infty xe^{-ax}\,dx = \frac{1}{a^2} \qquad (a > 0)$$
$$\int_0^\infty x^2e^{-ax}\,dx = \frac{2}{a^3} \qquad (a > 0)$$
$$\int_0^\infty x^ne^{-ax}\,dx = \frac{n!}{a^{n+1}} \qquad (a > 0, n > -1)$$
$$\int_0^\infty x^{1/2}e^{-ax}\,dx = \frac{1}{2a}\sqrt{\pi/a} \qquad (a > 0)$$
$$\int_0^\infty x^{-1/2}e^{-ax}\,dx = \sqrt{\pi/a} \qquad (a > 0)$$
$$\int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{b}{a^2 + b^2} \qquad (a > 0)$$
$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2 + b^2} \qquad (a > 0)$$
$$\int_0^\infty xe^{-ax}\sin(bx)\,dx = \frac{2ab}{(a^2 + b^2)^2} \qquad (a > 0)$$

$$\int_0^{\infty} xe^{-ax}\cos(bx)\,dx = \frac{a^2-b^2}{(a^2+b^2)^2} \qquad (a>0)$$

$$\int_0^{2\pi} \sin(ax)\,dx = 0 \qquad (a=1,2,3,\ldots)$$

$$\int_0^{2\pi} \cos(ax)\,dx = 0 \qquad (a=1,2,3,\ldots)$$

$$\int_0^{2\pi} \sin^2(ax)\,dx = \pi \qquad (a=1,2,3,\ldots)$$

$$\int_0^{2\pi} \cos^2(ax)\,dx = \pi \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \cos(ax)\,dx = 0 \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \sin(ax)\,dx = \frac{1}{a}[1-\cos(a\pi)] \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \sin^2(ax)\,dx = \frac{\pi}{2} \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \cos^2(ax)\,dx = \frac{\pi}{2} \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \sin(ax)\sin(bx) = 0 \qquad a\neq b \text{ } (a \text{ and } b \text{ are integers})$$

$$\int_0^{\pi} \cos(ax)\cos(bx) = 0 \qquad a\neq b \text{ } (a \text{ and } b \text{ are integers})$$

$$\int_0^{\pi} \sin(ax)\cos(bx) = 0 \qquad a=b \text{ } (a \text{ and } b \text{ are integers})$$

$$= 0 \qquad a\neq b \text{ but } (a+b) \text{ even}$$

$$= \frac{2a}{a^2-b^2} \qquad a\neq b \text{ but } (a+b) \text{ odd}$$

$$\int_0^{\pi/2} \sin(ax)\,dx = \frac{1}{a}[1-\cos(a\pi/2)]$$

$$\int_0^{\pi/2} \cos(ax)\,dx = \frac{1}{a}\sin(a\pi/2)$$

$$\int_0^{\pi/2} \sin^2(ax)\,dx = \frac{\pi}{4} \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi/2} \cos^2(ax)\,dx = \frac{\pi}{4} \qquad (a=1,2,3,\ldots)$$